Origametria and the van Hiele Theory of Teaching Geometry

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1 Introduction

The geometry curriculum implemented by the Israeli Ministry of Education follows closely the theories of the van Hieles [Fuys et al. 84, van Hiele 86, Burger and Shaughnessy 86]. In this paper, we show how the practices developed for the Origametria program can provide the solid foundation students need to prepare them to move into higher levels of abstraction in geometry.

Since 1992, the Israeli Origami Center (IOC) has trained teachers to teach the Origametria program in schools, and kindergarten Origametria in preschool. In 2010, after several years of close scrutiny, the Israeli Ministry of Education gave formal approval to the IOC to train preschool teachers for the kindergarten Origametria program. During the 2009–2010 academic year, the program operated nationwide in 35 primary schools (approximately 2.5% of all primary schools), half of which are Arab. The Origametria kindergarten and primary school programs are continually evolving.

The Origametria program teaches topics of curriculum geometry through the use of origami models. Unlike almost all other programs that teach geometry with origami, the models taught are not geometric subjects such as boxes, cubes, pyramids, modulars, and the like, but animals and action toys, which the students find fun and motivating. Also, the geometry of the final model is rarely, if ever analyzed. Instead, the geometry of the paper during the folding of the model is analyzed for its geometric content.

The wide experience of the IOC in running year-long programs in schools of different faiths, abilities, class numbers, and educational systems for almost two decades has enabled it to develop a distinctive style of classroom teaching which it considers integral to the growth and success of Origametria. Several schools have participated in the program for many years, citing that it helps the students attain better grades in the national TIMMS mathematics tests.

2 The van Hiele Theory of Geometric Teaching

The van Hiele theory was developed in the 1950s by two Dutch mathematics teachers, Pierre and Dina van Hiele [Fuys et al. 84]. The theory attempts to explain how students learn geometry and why many have difficulty with higher-level cognitive processes, especially when they are expected to give geometric proofs.

According to this theory, the development of the mathematical thought process, especially geometry, can be divided into five levels:

- level 0: visualization,
- level 1: analysis,
- level 2: abstraction,
- level 3: deduction,
- level 4: rigor.

Note that the levels are sometimes described as running from level 1 to level 5, creating some confusion as to which level is being discussed. There are also alternative names given to each level. A useful primer on the van Hiele theory can be found in [Burger and Shaughnessy 86].

3 Origami and the Van Hiele Theory

After the IOC established the Origametria program in schools, many parallels were found with the van Hiele method of teaching geometry. In Israel, many students learn geometry at the van Hiele Deductive Level (level 3) in middle and high schools, before they have established their knowledge at the earlier levels. These students are required to formulate proofs when they still cannot identify a side or an angle, cannot find a polygon within a polygon, or do not know basic geometric definitions.

The Origametria programs teach geometry to kindergarten and early primary school students first at the visual level (level 0) then later at the analysis level (level 1). There is no sudden jump from one level to another, but a gradual shift of emphasis from one level to the next. The Origametria program uses the first three levels of the van Hiele model, although most of the teaching focuses on levels 0 and 1. Following is an overview of the van Hiele levels and their comparison with Origametria:

- van Hiele level 0: Visualization
  - *van Hiele:* "At this level, students learn the names of many geometric terms and forms. They can identify geometric forms and understand the differences between them."
  - *Origametria:* In kindergarten Origametria and in early primary school, students are exposed to basic geometric terms and forms such as side, vertex, square, rectangle, and triangle. In every lesson, Origametria makes repeated reference to these terms and forms, so that a basic understanding is reinforced many times.
- van Hiele level 1: Analysis
  - *van Hiele:* "At this level, students can identify and analyze characteristics of geometric forms."
  - *Origametria:* In the process of folding a model, a geometric subject (such as an isosceles triangle or a line of symmetry) appears many times in different guises, and its character is discussed each time by the students. By this cumulative analysis of different examples, the characteristics of a geometric subject are learned.
- van Hiele level 2: Abstraction
  - *van Hiele:* "Students can understand the relationships and differences between polygons, and understand the importance of accurate definitions. Students can identify sets of shapes and their subsets (e.g., why all rectangles are in the family of parallelograms)."
  - *Origametria:* While folding a model, students test the characteristics of a geometric subject in different contexts, learning to separate and define similar-looking forms, such as scalene and isosceles triangles or parallelograms and rhombuses.

4 Time of Learning

The students understand what is being taught when it is within their time of learning, that is, when they are mature enough to understand the sub-

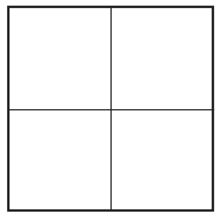


Figure 1. Find the number of squares in the figure.

ject. The class structure in Origametria generates an ambience that enables the student to arrive at the required time. This ambience is characterized by a constant analysis of the paper throughout the process of folding a model. The teacher helps the students to investigate selected geometric subjects with the students.

This process will enable each student to explore and learn the geometric subject at his/her own maturity and pace. The use of folding, identification of shapes, and investigation during the folding process will enable learning while actively and gradually building the knowledge at each student's pace. Even if at first the student only partially understands a concept, the repetitive investigation process while the paper is being folded will enable the student to gradually learn, but without feeling that he or she does not understand.

A good example of how students learn by analysis is shown in Figure 1, which asks how many squares the student can identify. At first, the students will say four; only later will they identify the fifth. A student within the time of learning will identify five squares, whereas a student who is not within this time will not see the fifth square. This latter student will learn from the answers of his or her peers and the teacher's explanations. The next time a similar question is posed, the student will immediately search and find the additional square. It is possible to extend this exercise, and ask how many quadrangles can be identified.

5 Gradually Building Knowledge and Concepts

Geometric terms are introduced to the students first using visual descriptions such as "side" or "vertex." This is the van Hiele level 0, where definitions are not taught, but are learned at an intuitive level of understanding. Later, at the van Hiele level 1, definitions are given, and the students' understanding will move from unconscious intuition to conscious knowledge. Origametria enables the same term to be used in many different circumstances at many different steps during the folding of a model, and from model to model. Thus, an intuitive understanding of a definition is built up and then confirmed when a definition is given.

One example is the diagonal of a polygon. A student can understand the definition of a diagonal of a polygon only after learning and understanding what the nonadjacent vertices are.[1] Thus, a student can grasp what a diagonal is only after having understood what the adjacent and nonadjacent vertices of a diagonal are. However, in doing origami, the student has already heard that term, at least for squares.

6 Using Origametria to Eliminate Misconceptions

During Origametria lessons in kindergarten and primary school, the students repeatedly experience creating and identifying polygons. This experience occurs in every lesson, enables the students to accumulate knowledge based on increasingly accurate intuition, and assists in eliminating misconceptions.

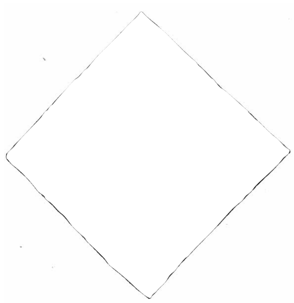
One example of a misconception is in the identification of a square. A familiar case is the one in which students identify and recognize a square in its familiar vertical-horizontal orientation, but when it is rotated 45 degrees, students no longer recognize it as a square but as a diamond. This occurs also with other polygons, such as isosceles triangles or right-angle triangles, where any rotation away from symmetry on the page, or from a polygon with a horizontal base, can lead to misidentification.

In Origametria lessons, the paper is folded into different polygons. The students investigate and learn to identify and define the polygons in different orientations during the natural rotation of the paper throughout the process of folding. This process enables them to avoid these misconceptions in their later studies.

7 Origametria and van Hiele: An Example from the Classroom

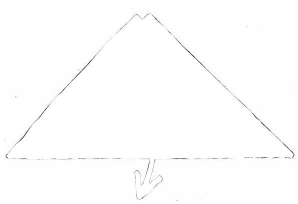
Below is an example of how an origami model—in this instance, the traditional Chinese duck—can be taught by the van Hiele–Origametria method (Figures 2–16). Depending on the level of the students, the questions asked are either at the visual level (van Hiele level 0) or at the analytic level (van Hiele level 1). The example is particularly appropriate to be taught in grades 1–3, although it may also be taught in other grades at the discretion of the teacher.

[1] A definition of a diagonal within a polygon is "a line segment linking two nonadjacent vertices" [Page 09].



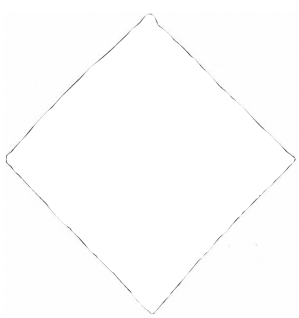
Visual level	What polygons can you find? What triangles can you identify? What kind of angles do you find in a square?
Analytic level	What is the sum of the angles in a square?

Figure 2. Step 1.



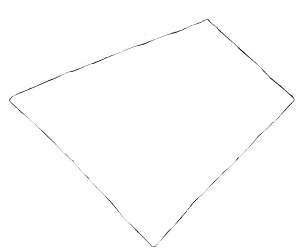
Visual level	What is the polygon? What triangles can you identify? What are the angles?
Analytic level	What is the sum of the angles in a triangle?

Figure 3. Step 2.



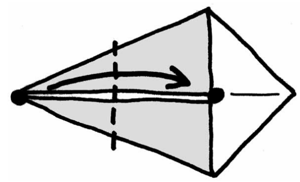
The paper is folded without analysis. In Origametria, not every folding step is examined for its geometric content. However, if this model is being used to teach bisections, this step would be discussed.

Figure 4. Step 3.



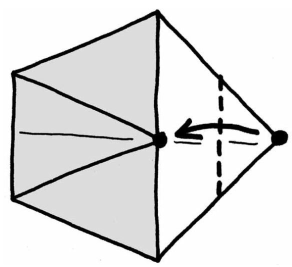
Visual level	What polygons can you find? What triangles can you identify? What kind of angles can you find?
Analytic level	What are the angles of the quadrilateral created after folding the step?

Figure 5. Step 4.



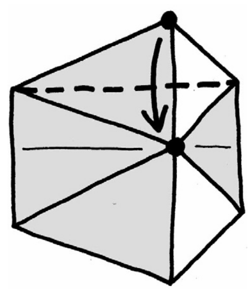
Visual level	What polygons can you find? What triangles can you identify? What kind of angles do you see in the polygons?
Analytic level	What are the angles in each corner?

Figure 6. Step 5.



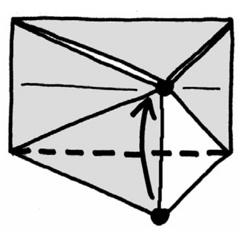
Visual level	What polygons can you find? What triangles can you identify? What kind of angles do you see in the pentagon?
Analytic level	What is the total number of degrees in a pentagon?

Figure 7. Step 6.



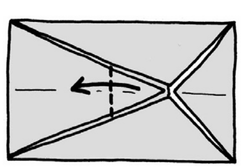
No questions.

Figure 8. Step 7.



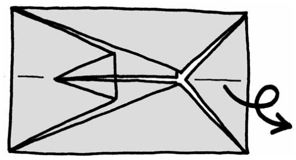
No questions.

Figure 9. Step 8.



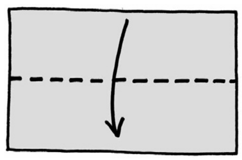
Visual level	What polygons can you find? What triangles can you identify? What kind of angles do you see?
Analytic level	Show that the total number of degrees where the four triangles meet is 360°.

Figure 10. Step 9.



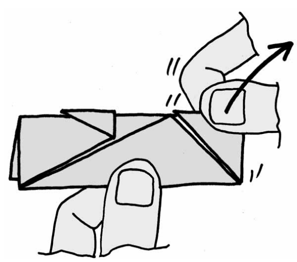
No questions.

Figure 11. Step 10.



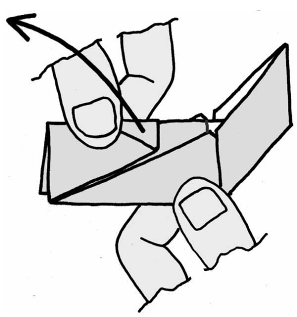
Visual level	How many rectangles can you find?

Figure 12. Step 11.



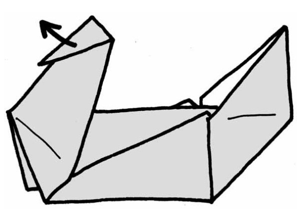
No questions.

Figure 13. Step 12.



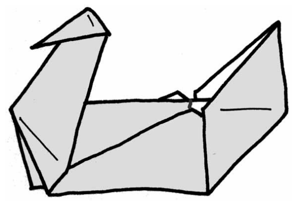
No questions.

Figure 14. Step 13.



No questions.

Figure 15. Step 14.



No questions.

Figure 16. Step 15.

The students are not told what model they are folding. This approach helps to abstract the paper so that a sharp protruding point can be identified as an isosceles triangle and not as a "leg," or whatever. It also frees a child's imagination to complete a model and to name it. The van Hieles also used play in the classroom, allowing children to play imaginatively with tangrams.

8 Conclusion: The Benefits of Using Origametria in the van Hiele System

It can be seen that Origametria supports the van Hiele method of teaching geometry, particularly at levels 0 and 1 and can offer benefits over traditional methods of teaching geometry. The van Hieles identified the lack of teaching of levels 0 and 1 as the main reason for poor performances in middle and high schools at levels 3 and 4. The focus of Origametria is on levels 0 and 1, helping to give students a strong foundation of geometric knowledge for performing successfully later at higher levels.

Making origami models in each lesson keeps the students' motivation to learn very high. Origami puts fun and fascination into learning topics that would otherwise be too dry and abstract for many children to enjoy learning. The van Hieles accepted that fun and creativity in a lesson motivated children to learn.

Using this approach, Origametria has helped less able children to improve. The constant repetition of topics in levels 0 and 1 helps all children to learn and to enter middle school better able to learn at higher levels of van Hiele. Further, there are many anecdotal reports from IOC teachers of children with learning difficulties or behavioral problems enjoying origami, succeeding in folding a model and thus, being motivated to learn more. For these children, the acquisition of geometric knowledge is incidental, but occurs nonetheless.

The main contribution of Origametria to the van Hiele theory at levels 0 and 1 is to help students better recognize and define terms and shapes fundamental to an understanding of geometry at higher levels. This ability is achieved by the constant rotation, turning over, manipulation and folding of the paper through a multiplicity of shapes, so that terms and shapes are identified many times, but each time in a unique context. This theme and variation approach to teaching strengthens each student's flexibility in thinking, ability to recognize and define, and at higher van Hiele levels, to deduce and extrapolate.

Origami is also suited for mathematical investigation at levels 3 and 4, though the IOC does not currently teach the Origametria program by the van Hiele method at these levels. Once Origametria is firmly established at the lower levels, we will build upon these foundations and expand into the higher levels.

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